Two-dimensional semi-infinite multicritical Ising models

1994 J. Phys. A: Math. Gen. 27 41

(http://iopscience.iop.org/0305-4470/27/1/004)

J. Phys. A: Math. Gen. 27 (1994) 41–54. Printed in the UK

Two-dimensional semi-infinite multicritical Ising models

Per Fröjdh†

Institute of Theoretical Physics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

Received 23 June 1993

Abstract. The order-parameter correlation functions of the $\mathbb{Z}_2$-invariant multicritical points in the unitary minimal series of conformal field theory are derived for a semi-infinite plane constrained to fixed or free boundary conditions. These yield the corresponding universal surface exponents which distinguish the behaviour between even- and odd-critical models. We also make explicit the interplay between duality and boundary conditions.

1. Introduction

The implications of conformal invariance of a statistical mechanical system at criticality are very powerful in two dimensions. All so-called minimal conformal field theories are completely classified, the scaling dimensions of their operators are determined and one knows in principle how to calculate all correlation functions [1, 2].

From extending this work to semi-infinite systems one also knows how to calculate correlation functions in critical systems with conformally invariant boundary conditions [3]. This has made it possible to determine surface exponents that govern correlations along a boundary of several statistical models including the Q-state Potts- and $O(N)$-models [3]. Moreover, the order-parameter correlation functions of the former have been constructed in a half-plane with fixed or free boundary conditions [4].

In this paper we go one step further by carrying out a detailed derivation of the order-parameter correlation functions in a semi-infinite plane, constrained to fixed or free boundary conditions, for a class of N-critical models that describe the simplest type of multicritical behaviour in a statistical system. These universality classes constitute the direct generalizations of the Ising and tricritical Ising models and span the whole series of unitary minimal models of conformal field theory [5]. It turns out that the behaviour of the order-parameter in a domain with a free boundary differs significantly between even- and odd-critical points, whereas for a fixed boundary it is essentially the same.

The rest of the paper is organized as follows. First we introduce the multicritical Ising models and identify the relevant operators for the following analysis. In section 3, we briefly review how one derives correlation functions in an infinite plane and present the four-point functions of the order-parameters of the models we are considering. Using these explicit solutions, we continue in section 4 with the central part of the paper, the derivation of the corresponding half-plane correlation functions for fixed and free boundary conditions. We also extract the surface exponents and conclude with a discussion on duality. In the last section we summarize our results and discuss related work and future extensions.

† Present address: Department of Physics, FM-15, University of Washington, Seattle, WA 98195, USA.

0305-4470/94/010041+14$07.50

2. Multicritical Ising models

In a Landau–Ginzburg classification of a multicritical point the free energy can have N degenerate minima corresponding to the co-existence of N different thermodynamic phases. For a certain choice of scaling fields these phases become identical, defining the N-critical point. This is described in two dimensions by the Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^2 r \left[(\nabla\varphi)^2 + g_1\varphi + g_2\varphi^2 + \cdots + g_{2N-2}\varphi^{2N-2} + g\varphi^{2N} \right] \tag{1}$$

where $\varphi(\boldsymbol{r})$ is a scalar order parameter, and the scaling fields g_i, $i = 1, 2, \ldots, 2N-2$ vanish at the N-critical point. The critical theories possess a $\mathbb{Z}_2$-invariance ($\varphi \to -\varphi$) and have been mapped to the unitary series of minimal models $\mathcal{M}_m$, $m = N + 1 = 3, 4, \ldots$, in conformal field theory [5].

The first member of the series is the ferromagnetic spin-$\frac{1}{2}$ Ising model ($m = 3$) where the critical point defines the transition from co-existence of two ferromagnetic phases to a paramagnetic phase. The next member, the tricritical Ising model, is a diluted Ising model with spins and vacant sites in thermodynamic equilibrium. Here the two ferromagnetic phases can co-exist with a magnetically disordered phase whose ground state consists of vacant sites only. The subsequent universality classes in the series are analogously labelled as higher-critical Ising models. All these $\mathbb{Z}_2$-invariant multicritical points have also been identified by Huse [6] with certain multicritical transitions in the restricted solid-on-solid (RSOS) models—exactly solved by Andrews *et al* in a subspace of the full parameter space [7].

The relevant operators of each model correspond to primary fields $\phi_{p,q}$ with scaling dimensions $x_{p,q} = 2h_{p,q}$ given by the Kac formula for conformal weights

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \qquad 1 \leqslant p \leqslant m-1 \qquad 1 \leqslant q \leqslant m . \tag{2}$$

The order parameter φ (magnetization σ in a ferromagnetic spin model) is identified with the most relevant non-trivial primary field

$$\phi_{2,2} = \varphi \qquad h_{2,2} = \frac{3}{4m(m+1)} . \tag{3}$$

The second relevant operator (energy density ϵ) is given by the renormalized composite field $:\varphi^2:$ defined as the leading operator in the operator–product expansion (OPE) of $\varphi\varphi - \langle\varphi\varphi\rangle$. It is even under spin reversal ($\sigma \to -\sigma$) and is identified with

$$\phi_{3,3} \simeq :\varphi^2: \qquad h_{3,3} = \frac{2}{m(m+1)} \tag{4}$$

for $m \geqslant 4$. Less relevant operators are similarly classified as spin- or energy-like, depending on their behaviour under spin reversal.

In the following, we will concentrate mainly on operators that appear in the OPE of $\sigma\sigma$. They belong to the even sector of the $\mathbb{Z}_2$-symmetry and are, in addition to $\phi_{1,1}$ (identity $\mathbb{1}$) and $\phi_{3,3}$,

$$\phi_{3,1} \simeq :\varphi^{2m-2}: \qquad h_{3,1} = 1 + \frac{2}{m} \tag{5a}$$

$$\phi_{1,3} \simeq :\varphi^{2m-4}: \qquad h_{1,3} = 1 - \frac{2}{m+1} . \tag{5b}$$

For the Ising model, however, $\phi_{3,1}$ is absent and $\phi_{1,3}$ is identified with the energy density ϵ. For the tricritical Ising model these energy-like operators are distinguished by even- and oddness under the Kramers–Wannier duality-transformation ($\epsilon \to -\epsilon$).

In addition to these operators, it is necessary to include two others in the analysis. Depending on m they are either even or odd under spin reversal and are given by

$$\phi_{2,1} \simeq: \varphi^{m-1} : \qquad h_{2,1} = \frac{m+3}{4m} \tag{6a}$$

$$\phi_{1,2} \simeq: \varphi^{m-2} : \qquad h_{1,2} = \frac{m-2}{4(m+1)} . \tag{6b}$$

3. Correlation functions in the plane

Any infinite-plane correlation function of primary fields $\phi_{p,q}$ in the minimal series $\mathcal{M}_m$ can be calculated by solving a partial differential equation of order pq. Hence, for the order parameters $\sigma = \phi_{2,2}$ in the multicritical Ising models, $\langle\sigma\cdots\sigma\rangle$ satisfies such an equation of order four. Here we will derive this equation and present its solution for the four-point function in a form suitable for the half-plane analysis in the next section.

A general method to derive an equation of this sort is to start by constructing a so-called null vector with zero norm [1]. The descendant states that can be generated from the primary state $|h\rangle = \phi_{2,2}|0\rangle$ by applying the Virasoro generators $L_n^\dagger = L_{-n}$ are degenerate at the fourth level, which means that the null vector $|\chi\rangle = \chi|0\rangle$ must have the form

$$|\chi\rangle = \mathcal{O}|h\rangle = \left(L_{-4} + c_1 L_{-1} L_{-3} + c_2 L_{-2}^2 + c_3 L_{-1}^2 L_{-2} + c_4 L_{-1}^4\right)|h\rangle . \tag{7}$$

Any correlation function of the corresponding null field χ and other primary fields will then be identical to zero. The parameters c_i, $i = 1, 2, 3$ and 4, are determined by two consistency equations $[L_1, \mathcal{O}]|h\rangle = 0$ and $[L_2, \mathcal{O}]|h\rangle = 0$ that can be solved for each $h = h_{2,2}$ in (3). It is then straightforward to verify that $\langle\chi|\chi\rangle = 0$. From the definition of a descendant field $L_{-n}\phi$ in terms of Fourier modes of the energy–momentum tensor acting on ϕ, it follows that [1]

$$\langle[L_{-n}\phi(z_1)]\phi(z_2)\cdots\phi(z_N)\rangle = \mathcal{L}_{-n}\langle\phi(z_1)\phi(z_2)\cdots\phi(z_N)\rangle \tag{8}$$

where $\mathcal{L}_{-n}$ is a differential operator: $\mathcal{L}_0 = h$, $\mathcal{L}_{-1} = \partial/\partial z_1$ and

$$\mathcal{L}_{-n} = -\sum_{i=2}^{N}\left[\frac{(1-n)h}{(z_i - z_1)^n} + \frac{1}{(z_i - z_1)^{n-1}}\frac{\partial}{\partial z_i}\right] \qquad n \geqslant 2 . \tag{9}$$

Using analogous relations for more complex descendant fields and the explicit form of the null field χ, one finds the correlation function $\langle\sigma\cdots\sigma\rangle$ to satisfy the fourth-order partial differential equation

$$\left\{\mathcal{L}_{-4} - \frac{15+4h}{6(3+h)}\mathcal{L}_{-1}\mathcal{L}_{-3} - \frac{4h}{9}\mathcal{L}_{-2}^2 + \frac{3+2h}{3(3+h)}\mathcal{L}_{-1}^2\mathcal{L}_{-2} - \frac{1}{4(3+h)}\mathcal{L}_{-1}^4\right\}\langle\sigma\cdots\sigma\rangle = 0 . \tag{10}$$

Invariance under the projective conformal group restricts the possible form of correlation functions [8] and thereby the solutions of the above equation. In particular, a four-point function may be written in terms of an unknown function of a single variable only:

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \frac{1}{(z_{13}z_{24})^{2h}}\frac{1}{[\rho(1-\rho)]^{2h}}F(\rho) \tag{11}$$

with $z_{ij} = z_i - z_j$ and the cross ratio

$$\rho = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \tag{12}$$

Applying the partial differential operator in (10) to this expression finally gives—after tedious but simple calculations—the following fourth-order ordinary differential equation:

$$\begin{aligned}\Bigg\{ & 16h^2\frac{6-16h+(3-32h+16h^2)\rho(\rho-1)}{\rho(\rho-1)} \\ & + \frac{(9-84h-16h^2(8h-15))\rho(\rho-1)-8h(3-8h)}{\rho(\rho-1)}2(2\rho-1)\frac{\mathrm{d}}{\mathrm{d}\rho} \\ & + 2(9+4h(8h-15)+(63+16h(11h-15))\rho(\rho-1))\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} \\ & + 12(3-4h)\rho(\rho-1)(2\rho-1)\frac{\mathrm{d}^3}{\mathrm{d}\rho^3} + 9\rho^2(\rho-1)^2\frac{\mathrm{d}^4}{\mathrm{d}\rho^4}\Bigg\}F(\rho) = 0\end{aligned} \tag{13}$$

with h given by (3) for each $m = 3, 4, \ldots$ of $\mathcal{M}_m$. We have used this equation to verify† the known correlation function $\langle\sigma\sigma\sigma\sigma\rangle$ in [5]. Including some prefactors for later convenience, the four solutions are

$$\begin{aligned} f_{11}(m|\rho) = {} & 2(1-\rho)F\left(1-\frac{3}{m}, 1-\frac{1}{m}; 1-\frac{2}{m}; \rho\right) F\left(\frac{1}{m+1}, \frac{3}{m+1}; \frac{2}{m+1}; \rho\right) \\ & + \frac{\rho}{m-2}F\left(1-\frac{3}{m}, 1-\frac{1}{m}; 2-\frac{2}{m}; \rho\right) F\left(\frac{1}{m+1}, \frac{3}{m+1}; 1+\frac{2}{m+1}; \rho\right)\end{aligned} \tag{14a}$$

$$\begin{aligned} f_{31}(m|\rho) = {} & 2(-\rho)^{2/m}\Bigg\{(1-\rho)F\left(1+\frac{1}{m}, 1-\frac{1}{m}; 1+\frac{2}{m}; \rho\right) \\ & \times F\left(\frac{1}{m+1}, \frac{3}{m+1}; \frac{2}{m+1}; \rho\right) \\ & - F\left(\frac{1}{m}, -\frac{1}{m}; \frac{2}{m}; \rho\right) F\left(\frac{1}{m+1}, \frac{3}{m+1}; 1+\frac{2}{m+1}; \rho\right)\Bigg\}\end{aligned} \tag{14b}$$

† Of the four functions in [5], the one given by (A5*c*) does not satisfy our equation (13). However, it also fails to produce the Ising-model result ($m = 3$, $h = \frac{1}{16}$) which can be calculated in a simpler fashion [1]. Guided by this special case, we have corrected the corresponding solution and verified that it satisfies (13) for other values of m as well.

$$f_{13}(m|\rho) = (-\rho)^{1-2/(m+1)}\left\{\frac{1}{m-2}F\left(1-\frac{3}{m},1-\frac{1}{m};2-\frac{2}{m};\rho\right)\right.$$
$$\times F\left(\frac{1}{m+1},-\frac{1}{m+1};1-\frac{2}{m+1};\rho\right)$$
$$-\frac{(1-\rho)}{m-1}F\left(1-\frac{3}{m},1-\frac{1}{m};1-\frac{2}{m};\rho\right)$$
$$\left.\times F\left(1+\frac{1}{m+1},1-\frac{1}{m+1};2-\frac{2}{m+1};\rho\right)\right\} \tag{14c}$$

$$f_{33}(m|\rho) = (-\rho)^{2/[m(m+1)]}\left\{2F\left(\frac{1}{m},-\frac{1}{m};\frac{2}{m};\rho\right)F\left(\frac{1}{m+1},-\frac{1}{m+1};1-\frac{2}{m+1};\rho\right)\right.$$
$$+\frac{\rho(1-\rho)}{m-1}F\left(1+\frac{1}{m},1-\frac{1}{m};1+\frac{2}{m};\rho\right)$$
$$\left.\times F\left(1+\frac{1}{m+1},1-\frac{1}{m+1};2-\frac{2}{m+1};\rho\right)\right\} \tag{14d}$$

with $F(a,b;c;\rho)$ a hypergeometric function. The labels of the solutions refer to the corresponding operators $\phi_{p,q}$ in the OPE

$$\phi_{2,2}\phi_{2,2} \sim \sum_{p,q=1,3} C_{pq}\phi_{p,q} \tag{15}$$

which defines the structure constants C_{pq}.

So far we have concentrated on the analytic dependence of $\langle\sigma\sigma\sigma\sigma\rangle$. Conformal invariance implies that the anti-analytic dependence satisfies an analogous equation for $\bar{\rho}$. Hence, the complete expression for the infinite-plane correlation function is a combination of such solutions:

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{13}z_{24}\rho(1-\rho)|^{4h}}\sum_{p,q=1,3} S_{pq}f_{pq}(m|\rho)\,f_{pq}(m|\bar{\rho}) \tag{16}$$

where the coefficients S_{pq} are determined by requiring $\langle\sigma\sigma\sigma\sigma\rangle$ to be single-valued. By normalizing the functions $f_{pq}(m|\rho)$ it is also possible to determine (the squares of) the structure constants in (15). However, all such structure constants for the minimal theories are already known [9], and in particular [5]

$$C_{11}^2 = 1 \tag{17a}$$

$$C_{31}^2 = \frac{3}{4(m+2)^2(m+3)^2}\left\{\left[\Gamma\left(1+\frac{1}{m}\right)\Gamma^2\left(1-\frac{2}{m}\right)\Gamma\left(1+\frac{3}{m}\right)\right]\right.$$
$$\left.\Big/\left[\Gamma\left(1-\frac{1}{m}\right)\Gamma^2\left(1+\frac{2}{m}\right)\Gamma\left(1-\frac{3}{m}\right)\right]\right\} \tag{17b}$$

$$C_{13}^2 = \frac{3}{4(m-1)^2(m-2)^2}\left\{\left[\Gamma\left(1-\frac{1}{m+1}\right)\Gamma^2\left(1+\frac{2}{m+1}\right)\Gamma\left(1-\frac{3}{m+1}\right)\right]\right.$$
$$\left.\Big/\left[\Gamma\left(1+\frac{1}{m+1}\right)\Gamma^2\left(1-\frac{2}{m+1}\right)\Gamma\left(1+\frac{3}{m+1}\right)\right]\right\} \tag{17c}$$

$$C_{33}^2 = \left[\Gamma\left(1+\frac{1}{m}\right)\Gamma^2\left(1-\frac{2}{m}\right)\Gamma\left(1+\frac{3}{m}\right)\Gamma\left(1-\frac{1}{m+1}\right)\Gamma^2\left(1+\frac{2}{m+1}\right)\Gamma\left(1-\frac{3}{m+1}\right)\right]$$
$$\Bigg/\left[\Gamma\left(1-\frac{1}{m}\right)\Gamma^2\left(1+\frac{2}{m}\right)\Gamma\left(1-\frac{3}{m}\right)\Gamma\left(1+\frac{1}{m+1}\right)\right.$$
$$\left.\times\,\Gamma^2\left(1-\frac{2}{m+1}\right)\Gamma\left(1+\frac{3}{m+1}\right)\right]. \tag{17d}$$

As will be seen in the following section on half-plane correlation functions, the previous set of solutions (14), as well as the structure constants (17), are also the basic ingredients for determining order-parameter correlations in bounded domains with fixed and free boundary conditions.

4. Semi-infinite geometry and boundary conditions

The semi-infinite plane is the simplest geometry for studying boundary effects in a two-dimensional model. Using conformal transformations to other domains, such as an infinite strip, disc or square, many results may be directly carried over to these geometries. In particular, the half-plane correlation functions can be mapped to all other simply-connected domains so that surface exponents, for instance, can be determined.

Introducing a boundary to a system allows new types of transitions to occur in its neighbourhood [10]. At the so-called *ordinary transition* the boundary is free (open) and orders simultaneously with the bulk. The corresponding boundary condition in the continuum limit is that the magnetization vanishes at the boundary: $\langle\sigma\rangle = 0$. At the *extraordinary transition* the boundary spins are fixed in one direction so that the bulk orders in the presence of an ordered boundary. This is referred to as a fixed boundary condition, for which the magnetization diverges at the boundary in the continuum limit: $\langle\sigma\rangle \to \infty$. In dimensions higher than two, there is also the possibility that the surface orders independently without applying an external field.

The former techniques for deriving correlation functions in the infinite plane can now also be used for a theory defined in a half-plane [3]. Due to the geometric constraint of the latter, half of the conformal symmetries are redundant so that one is left with only analytic functions. As a result, an n-point function in the upper half-plane will satisfy the same differential equation as a $2n$-point function in the plane. However, the solutions are combined in a different manner in order to respect the appropriate boundary conditions. For a correlation function whose solutions satisfy a second-order differential equation, the two solutions themselves provide enough information for such a choice. In the case we are considering here, we have four solutions, and in order to construct the right combination for each boundary condition, we have to make a detailed analysis of the bulk properties of the wanted correlation function $\langle\sigma\sigma\rangle$ as well as the boundary properties of the fields in the operator-product algebra of $\sigma\sigma$.

Due to the presence of a boundary, correlations will decay with different exponents parallel and perpendicular to it. The power-law correlation between two points parallel to the boundary,

$$\langle\phi(z_1)\phi(z_2)\rangle_c = \langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle \sim \frac{1}{|z_1 - z_2|^{2x_\parallel}} \qquad \text{as} \quad |z_1 - z_2| \to \infty \tag{18}$$

defines the *surface exponent* $x_\parallel$ ($2x_\parallel = \eta_\parallel$). For any direction not parallel to the boundary, i.e. when one of the points approaches the bulk, the analogous exponent is governed by

$(x + x_\parallel)/2$. Hence, $x_\parallel$ is naturally interpreted as a *boundary scaling dimension*. It appears in the Kac formula (2) for boundary operators and is universal for each model and boundary transition. However, a boundary operator does not in general have the same (p, q)-indices as the corresponding primary field in the bulk. It need not even be primary, i.e. it can have $x_\parallel = h_{p,q} + n$, where n is a positive integer. From the half-plane construction of conformal field theory one can see that the boundary operators are found in the OPEs of bulk operators and their mirror images on the other side of the boundary. Hence, $x_\parallel$ corresponding to a bulk operator ϕ is to be found in the OPE of ϕ with itself.

After these general remarks, let us now turn to the calculation of half-plane correlation functions. We will derive $\langle\sigma\sigma\rangle_c$ for free and fixed boundary conditions on the order parameter σ. Moreover, their asymptotic behaviour will determine the boundary dimensions $x_\parallel$ that, together with the scaling dimensions $x = 2h_{2,2}$ in (3), determine all other boundary dimensions in any two-dimensional geometry. In addition, we will derive one-point amplitudes of energy-like operators (even under spin reversal) and directly see how duality relates fixed and free boundary conditions.

The relation between $\langle\sigma(z_1)\sigma(z_2)\rangle$ in the half-plane and $\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$ in the infinite plane is that they satisfy the same partial differential equation for $z_3 = \bar{z}_1$ and $z_4 = \bar{z}_2$ [3]. With this substitution in (11), we have

$$\langle\sigma\sigma\rangle = \frac{1}{(4y_1y_2)^{2h}} Y(\rho) \tag{19}$$

with

$$Y(\rho) = \frac{1}{[\rho(\rho-1)]^{2h}} \sum_{p,q=1,3} \alpha_{pq}\, f_{pq}(m|\rho) \tag{20}$$

and the cross-ratio

$$\rho = -\frac{(x_1-x_2)^2 + (y_1-y_2)^2}{4y_1y_2}. \tag{21}$$

The coordinates in the half-plane are given by $z_j = x_j + \mathrm{i}y_j$, $y > 0$. In contrast with the former case, this correlation function is single-valued for any choice of coefficients α_{pq}. The appropriate combination is instead determined by imposing free or fixed boundary conditions on the order parameter σ, which is most conveniently done by changing to a new basis for the solutions to (13). In the present basis (14), the hypergeometric functions can be represented by Gauss series for $|\rho| < 1$, i.e. in the bulk limit $|z_1 - z_2| \to 0$. However, as we expect to observe the influence of the boundary at long distances, $|z_1 - z_2| \to \infty$, we analytically continue (13) to $\rho = -\infty$ and obtain the following set that can be series expanded for $|\rho| > 1$:

$$\begin{aligned} g_{11}(m|\rho) = (-\rho)^{3/[m(m+1)]}\bigg\{ &\left(1-\frac{1}{\rho}\right) F\left(1-\frac{3}{m}, 1-\frac{1}{m}; 1-\frac{2}{m}; \frac{1}{\rho}\right) \\ &\times F\left(\frac{3}{m+1}, 1+\frac{1}{m+1}; 1+\frac{2}{m+1}; \frac{1}{\rho}\right) \\ &+ F\left(1-\frac{3}{m}, -\frac{1}{m}; 1-\frac{2}{m}; \frac{1}{\rho}\right) F\left(\frac{1}{m+1}, \frac{3}{m+1}; 1+\frac{2}{m+1}; \frac{1}{\rho}\right)\bigg\} \end{aligned} \tag{22a}$$

$$g_{31}(m|\rho) = (-\rho)^{(1-2m)/[m(m+1)]}\left\{\left(1-\frac{1}{\rho}\right) F\left(1+\frac{1}{m}, 1-\frac{1}{m}; 1+\frac{2}{m}; \frac{1}{\rho}\right)\right.$$
$$\times F\left(\frac{3}{m+1}, 1+\frac{1}{m+1}; 1+\frac{2}{m+1}; \frac{1}{\rho}\right)$$
$$\left.- F\left(1-\frac{1}{m}, \frac{1}{m}; 1+\frac{2}{m}; \frac{1}{\rho}\right) F\left(\frac{1}{m+1}, \frac{3}{m+1}; 1+\frac{2}{m+1}; \frac{1}{\rho}\right)\right\} \tag{22b}$$

$$g_{13}(m|\rho) = (-\rho)^{(2m+3)/[m(m+1)]}\left\{\left(1-\frac{1}{\rho}\right) F\left(1-\frac{3}{m}, 1-\frac{1}{m}; 1-\frac{2}{m}; \frac{1}{\rho}\right)\right.$$
$$\times F\left(\frac{1}{m+1}, 1-\frac{1}{m+1}; 1-\frac{2}{m+1}; \frac{1}{\rho}\right) - F\left(1-\frac{3}{m}, -\frac{1}{m}; 1-\frac{2}{m}; \frac{1}{\rho}\right)$$
$$\left.\times F\left(\frac{1}{m+1}, -\frac{1}{m+1}; 1-\frac{2}{m+1}; \frac{1}{\rho}\right)\right\} \tag{22c}$$

$$g_{33}(m|\rho) = (-\rho)^{1/[m(m+1)]}\left\{\left(1-\frac{1}{\rho}\right) F\left(1+\frac{1}{m}, 1-\frac{1}{m}; 1+\frac{2}{m}; \frac{1}{\rho}\right)\right.$$
$$\times F\left(\frac{1}{m+1}, 1-\frac{1}{m+1}; 1-\frac{2}{m+1}; \frac{1}{\rho}\right)$$
$$\left.+ F\left(1-\frac{1}{m}, \frac{1}{m}; 1+\frac{2}{m}; \frac{1}{\rho}\right) F\left(\frac{1}{m+1}, -\frac{1}{m+1}; 1-\frac{2}{m+1}; \frac{1}{\rho}\right)\right\}. \tag{22d}$$

The matrix that relates the two sets of solutions factorizes in the Kac indices:

$$f_{pq}(m|\rho) = M_{pr}(m) N_{qs}(m) g_{rs}(m|\rho)\,. \tag{23}$$

Here summation over repeated indices is implied, and

$$M(m) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad N(m) = M^T(m+1) \tag{24}$$

with

$$a = \frac{2\Gamma(2/m)\Gamma(-2/m)}{\Gamma(1/m)\Gamma(-1/m)} \tag{25a}$$

$$b = \frac{2\Gamma^2(-2/m)}{3\Gamma(-1/m)\Gamma(-3/m)} \tag{25b}$$

$$c = \frac{2\Gamma^2(2/m)}{\Gamma(1/m)\Gamma(3/m)}\,. \tag{25c}$$

Expressed in this new basis,

$$Y(\rho) = \frac{1}{[\rho(\rho-1)]^{2h}} \sum_{p,q=1,3} \beta_{pq} g_{pq}(m|\rho) \tag{26}$$

from which it follows that

$$\beta_{pq} = \alpha_{rs}\, M_{rp}(m)\, N_{sq}(m)\,. \tag{27}$$

The different boundary conditions together with the long-distance behaviour of the correlation function put restrictions on the coefficients β_{pq}. These restrictions can be determined from an expansion of (26) in inverse powers of ρ

$$Y(\rho) = \frac{\beta_{11}}{(-\rho)^{h_{1,1}}}\left[2 + \mathrm{O}\left(\frac{1}{\rho^2}\right)\right] + \frac{\beta_{31}}{(-\rho)^{h_{3,1}}}\left[\frac{3}{(m+2)(m+3)} + \mathrm{O}\left(\frac{1}{\rho}\right)\right] + \frac{\beta_{33}}{(-\rho)^{h_{3,3}}}\left[2 + \mathrm{O}\left(\frac{1}{\rho^2}\right)\right] + \frac{\beta_{13}}{(-\rho)^{h_{1,3}}}\left[\frac{1}{(m-1)(m-2)} + \mathrm{O}\left(\frac{1}{\rho}\right)\right] \tag{28}$$

with $h_{1,1} = 0$ and $h_{p,q}$ in (4) and (5). In the following discussion we will also use the fact that $0 < h_{3,3} < h_{1,3} < h_{3,1} < 2$ for $m \geqslant 3$.

Let us first consider fixed boundary conditions for which $\langle\sigma\rangle \neq 0$. Then, at large distances ($\rho \to -\infty$), we must have $\langle\sigma\sigma\rangle \to \langle\sigma\rangle\langle\sigma\rangle \neq 0$. Comparing this with (28), we conclude that $\beta_{11} \neq 0$. From the connected two-point function $\langle\sigma\sigma\rangle_c$ it is now possible to obtain the boundary dimension $x_\parallel$ (18). It has been shown by Burkhardt and Cardy that this must be an even integer as σ has a non-vanishing expectation value induced by the boundary [11]. Subtracting the disconnected part from (28) and comparing this with the analogous $Y_c(\rho) \sim \rho^{-x_\parallel}$, we see that in order to get an even integer, we must let $\beta_{31} = \beta_{13} = \beta_{33} = 0$. The value of β_{11} is finally fixed by requiring that the correlation function is normalized: $\langle\sigma\sigma\rangle \to |z_1 - z_2|^{-4h}$ as $|z_1 - z_2| \to 0$. By that we conclude for fixed boundary conditions

$$\beta_{11} = 2\cos\frac{\pi}{m}\cos\frac{\pi}{m+1} \tag{29a}$$

$$\beta_{31} = \beta_{13} = \beta_{33} = 0 \tag{29b}$$

$$x_\parallel = 2\,. \tag{29c}$$

Turning to the case of a free boundary, it is not as straightforward to select the appropriate solutions. Due to unbroken spin-reversal symmetry, $\langle\sigma\rangle = 0$, the two-point function decays at long distances: $\langle\sigma\sigma\rangle \to 0$. From this consideration we have that $\beta_{11} = 0$, but no other information about the coefficients in (28). Instead, we have to analyse the bulk limit $|z_1 - z_2| \to 0$, using the first set of solutions (14). From this we will derive equations for α_{pq} that in turn will impose more constraints on β_{pq}. However, it turns out to be advantageous to use this approach without specifying the boundary conditions from the beginning. By specializing to a fixed boundary at a later stage, this will make it possible to verify (29). It will also be easier to analyse the implications of a free boundary when we first have made analogous calculations for the simpler fixed-boundary case.

Taking the expectation value of the OPE (15) with the explicit co-ordinate dependence inserted, we obtain the following short-distance behaviour of $\langle\sigma\sigma\rangle$:

$$\langle\sigma(z_1)\sigma(z_2)\rangle = \frac{1}{|z_{12}|^{4h_{2,2}}}\sum_{p,q=1,3} |z_{12}|^{2h_{p,q}}\, C_{pq}\, \langle\phi_{p,q}(z_2)\rangle \;+\cdots\,. \tag{30}$$

The expectation values $\langle\phi\rangle$ follow from the infinite-plane correlation function $\langle\phi(z_1)\phi(z_2)\rangle \sim |z_1 - z_2|^{-2h}$ by substituting $z_2 = \bar{z}_1$:

$$\langle\phi(y)\rangle = \frac{\mathcal{A}}{(2y)^{2h}} \tag{31}$$

with $\mathcal{A}$ an amplitude depending on the boundary condition on σ. Using (19), we now see that

$$Y(\rho) = \frac{1}{(-\rho)^{2h_{2,2}}} \sum_{p,q=1,3} (-\rho)^{h_{p,q}} C_{pq} \mathcal{A}_{p,q} + \cdots \tag{32}$$

as $\rho \to 0$. As this expression has to coincide with a short-distance expansion of (20) analogous to (28), we are able to identify the corresponding coefficients

$$\alpha_{11} = \tfrac{1}{2} \tag{33a}$$

$$\alpha_{31} = \tfrac{1}{3}(m+2)(m+3)C_{31}\mathcal{A}_{31} \tag{33b}$$

$$\alpha_{13} = (m-1)(m-2)C_{13}\mathcal{A}_{13} \tag{33c}$$

$$\alpha_{33} = \tfrac{1}{2}C_{33}\mathcal{A}_{33} \tag{33d}$$

where we have explicitly used the normalization of the identity operator: $\mathcal{A}_{11} = 1$. The structure constants C_{pq} are determined only up to a sign and we are free to choose the positive solutions to (17). (One can always use non-negative structure constants in the unitary minimal series of conformal field theory [9].) If we now combine (33) with (27), we may eliminate the α_{pq}-coefficients and get four linear equations for β_{pq} including the three amplitudes $\mathcal{A}_{31}$, $\mathcal{A}_{13}$ and $\mathcal{A}_{33}$. At least one of the former is given by the previous conditions on β_{pq} and two of the latter can be determined independently, as will be shown in the following. Altogether, we will have four linear equations in four variables. It is then an easy task to solve these equations and write down the half-plane correlation functions for free and fixed boundaries.

First we need to determine the amplitudes $\mathcal{A}_{31}$ and $\mathcal{A}_{13}$. The corresponding operators do not only appear in the operator algebra we are considering here, but also in

$$\phi\phi \sim \mathbb{1} + C'\phi' \tag{34}$$

where $\phi = \phi_{1,2}(\phi_{2,1})$ and $\phi' = \phi_{1,3}(\phi_{3,1})$. Hence, we may extract the amplitudes from the half-plane correlation functions $\langle\phi_{1,2}\phi_{1,2}\rangle$ and $\langle\phi_{2,1}\phi_{2,1}\rangle$ that have been derived for all multicritical Ising models [12]. Comparing the expectation value of (34) with a short-distance expansion of the correlation functions in [12] we identify for $\langle\phi\rangle \neq 0$

$$C'\mathcal{A} = \frac{\Gamma((2+4h)/3))\Gamma(-(1+8h)/3)}{\Gamma(-4h)\Gamma((1+8h)/3)} \tag{35}$$

and for $\langle\phi\rangle = 0$

$$C'\mathcal{A} = \frac{\Gamma(1+4h)\Gamma(-(1+8h)/3)}{\Gamma((1-4h)/3)\Gamma((1+8h)/3)}. \tag{36}$$

Here the conformal weight h equals $h_{1,2}$ or $h_{2,1}$ in (6) and the structure constant C' is determined by the single-valuedness condition of the corresponding four-point functions in the infinite plane [13]. In both cases this gives

$$C' = \sqrt{\frac{4h\Gamma(1+4h)\Gamma((2+4h)/3)\Gamma^2(-(1+8h)/3)}{\Gamma(1-4h)\Gamma((1-4h)/3)\Gamma^2((1+8h)/3)}}. \tag{37}$$

The next step is to consistently combine (35) and (36) with the boundary conditions on the order parameter σ. As the fixed-boundary case is somewhat simpler; we start with this and compare the result with (29). In this case (35) applies for both $\phi = \phi_{1,2}$ and $\phi_{2,1}$†. Together with (37), using (6), we get

$$\mathcal{A}_{31} = \mathcal{A}(m) \tag{38a}$$

$$\mathcal{A}_{13} = \mathcal{A}(m+1) \tag{38b}$$

with

$$\mathcal{A}(m) = \sqrt{\frac{\Gamma(1/m)\Gamma(-1/m)}{3\Gamma(3/m)\Gamma(-3/m)}} = \sqrt{4\cos^2\frac{\pi}{m} - 1}\,. \tag{39}$$

As a check, we have verified this result by deriving it directly from (33) and (27) with the fixed-boundary condition $\beta_{11} = \beta_{13} = \beta_{31} = 0$. For later purposes we also give the third amplitude: $\mathcal{A}_{33} = \mathcal{A}_{31}\mathcal{A}_{13}$.

For a free boundary, we may have either of the two equations (35) and (36). To determine $\mathcal{A}_{13}$, on the one hand, we have to consider $\phi = \phi_{1,2}$ which is even (odd) under spin reversal for m even (odd). Hence, we must use (35)) for m even and (36) for m odd. For $\mathcal{A}_{13}$, on the other hand, the converse is true. Together with (37), this leads to

$$\mathcal{A}_{31} = \begin{cases} -\mathcal{A}^{-1}(m) & m \text{ even} \\ \mathcal{A}(m) & m \text{ odd} \end{cases} \tag{40a}$$

$$\mathcal{A}_{13} = \begin{cases} \mathcal{A}(m+1) & m \text{ even} \\ -\mathcal{A}^{-1}(m+1) & m \text{ odd}\,. \end{cases} \tag{40b}$$

Using these amplitudes in (33) combined with (27) and the condition $\beta_{11} = 0$, we finally obtain for a free boundary that all β_{pq} except β_{31} (β_{13}) vanish for m even (odd). We also have for the last amplitude $\mathcal{A}_{33} = \mathcal{A}_{31}\mathcal{A}_{13}$, as for the case of fixed boundary spins. Finally we extract the boundary dimension from the long-distance behaviour of $\langle\sigma\sigma\rangle$: $x_\parallel = h_{3,1}$ ($h_{1,3}$) for m even (odd). Altogether, for free boundary conditions and m even,

$$\beta_{31} = 2\cos\frac{\pi}{m}\cos\frac{\pi}{m+1}\frac{\Gamma(-2/m)\Gamma(3/m)}{\Gamma(2/m)\Gamma(-1/m)} \tag{41a}$$

$$\beta_{11} = \beta_{13} = \beta_{33} = 0 \tag{41b}$$

$$x_\parallel = 1 + \frac{2}{m} \tag{41c}$$

while for m odd,

$$\beta_{13} = 6\cos\frac{\pi}{m}\cos\frac{\pi}{m+1}\frac{\Gamma(2/(m+1))\Gamma(-3/(m+1))}{\Gamma(-2/(m+1))\Gamma(1/(m+1))} \tag{42a}$$

$$\beta_{11} = \beta_{31} = \beta_{33} = 0 \tag{42b}$$

$$x_\parallel = 1 - \frac{2}{m+1}\,. \tag{42c}$$

† In [12], $\langle\phi\phi\rangle$ with ϕ even ($\langle\phi\rangle \neq 0$) and ϕ odd ($\langle\phi\rangle = 0$) is derived for a free boundary. However, the first case ($\langle\phi\rangle \neq 0$) also applies to a fixed boundary for any ϕ ($= \phi_{1,2}$ or $\phi_{2,1}$).

For the Ising model ($m = 3$) we recover the known free-boundary correlation function and surface exponent $x_\parallel = \frac{1}{2}$ [3], and for the tricritical Ising model ($m = 4$) we confirm that $x_\parallel = \frac{3}{2}$. The latter was first obtained from analysing the operator content of the model as restricted by modular invariance and free boundaries [14]. It has also been derived by studying the finite-size scaling spectrum [15, 16]. It is interesting to notice that the effect of an external field coupling to the order parameter at the boundary is quite different for even and odd m. For the tri-, penta-, ... critical points we have $x_\parallel > 1$, i.e. the order parameter is irrelevant on the (one-dimensional) boundary. For the bi-, tetra-, ... critical points the converse is true, so that such an external field is relevant.

In summary we have derived the half-plane order-parameter correlation functions of the $(m-1)$-critical Ising models $\mathcal{M}_m$ ($m = 3, 4, \ldots$). For a fixed boundary condition on the order parameter, the connected correlation function is

$$\langle\sigma\sigma\rangle_c = \frac{2\cos\pi/m\cos\pi/(m+1)}{(4y_1y_2)^{3/[2m(m+1)]}}\left\{\frac{g_{11}(m|\rho)}{[\rho(\rho-1)]^{3/[2m(m+1)]}} - 2\right\} \tag{43}$$

yielding the boundary dimension $x_\parallel = 2$. For a free boundary, we have for m even

$$\langle\sigma\sigma\rangle = \left\{\left[2\cos\frac{\pi}{m}\cos\frac{\pi}{m+1}\frac{\Gamma(-2/m)\Gamma(3/m)}{\Gamma(2/m)\Gamma(-1/m)}\right]\Big/(4y_1y_2)^{3/[2m(m+1)]}\right\} \times \frac{g_{31}(m|\rho)}{[\rho(\rho-1)]^{3/[2m(m+1)]}} \tag{44}$$

and for m odd

$$\langle\sigma\sigma\rangle = \left\{\left[6\cos\frac{\pi}{m}\cos\frac{\pi}{m+1}\frac{\Gamma(2/(m+1))\Gamma(-3/(m+1))}{\Gamma(-2/(m+1))\Gamma(1/(m+1))}\right]\Big/(4y_1y_2)^{3/[2m(m+1)]}\right\} \times \frac{g_{13}(m|\rho)}{[\rho(\rho-1)]^{3/[2m(m+1)]}} \tag{45}$$

with the corresponding boundary dimensions

$$x_\parallel = \begin{cases} 1 + 2/m & m \text{ even} \\ 1 - 2/(m+1) & m \text{ odd}. \end{cases} \tag{46}$$

The functions $g_{pq}(m|\rho)$ are given by (22) and the co-ordinate dependence of ρ by (21).

As a by-product of this analysis, we have also determined a set of one-point amplitudes (defined for non-negative structure constants and such that all two-point functions are normalized to $|z_1 - z_2|^{-2x}$ as $|z_1 - z_2| \to 0$). For the order parameter $\sigma = \phi_{2,2}$ in the presence of a free boundary, the amplitude $\mathcal{A}$ must vanish in (31). However, for a fixed boundary condition that breaks the $\mathbb{Z}_2$-symmetry we have $\langle\sigma\rangle \neq 0$. The corresponding amplitude can be deduced from (43):

$$\mathcal{A}_\sigma = \pm 2\sqrt{\cos\frac{\pi}{m}\cos\frac{\pi}{m+1}} \tag{47}$$

where the sign reflects the direction of the order parameter at the boundary.

For the energy-like operators ϕ_{31} and ϕ_{13} we have derived amplitudes for fixed (38) and free (40) boundaries. In both cases, we also have that $\mathcal{A}_{33} = \mathcal{A}_{31}\mathcal{A}_{13}$. The Ising model includes only one of these energy-like operators, namely the energy density $\epsilon = \phi_{1,3}$. For a fixed boundary it yields $\mathcal{A}_\epsilon = 1$ and for a free boundary $\mathcal{A}_\epsilon = -1$. Hence, we see directly that fixed and free boundaries are related via the Kramers–Wannier duality transformation ($\epsilon \to -\epsilon$) [17, 18]. The tricritical Ising model has three energy-like operators which are distinguished by duality [19]. From (4) and (5) with $m = 4$ it follows that the energy density ($\epsilon = \phi_{3,3}$) and the second sub-leading energy operator ($\epsilon'' = \phi_{3,1}$) are dual odd, whereas the sub-leading energy operator ($\epsilon' = \phi_{1,3}$) is dual even. This is verified by the fact that the fixed-boundary amplitudes $\mathcal{A}_\epsilon = \mathcal{A}_{\epsilon'} = \sqrt{2/(\sqrt{5}-1)}$ and $\mathcal{A}_{\epsilon''} = 1$ transform as $\mathcal{A} \to -\mathcal{A}$ for ϵ and ϵ'', and $\mathcal{A} \to \mathcal{A}$ for ϵ' when changing to free boundaries. Once again we see that the duality transformation relates fixed and free boundaries.

For the other models ($m \geqslant 5$) we have a different scenario. It follows from (38) and (40) that it is not possible to interchange fixed and free boundaries by just changing sign of 'dual odd' operators. It is also necessary to alter the magnitude, as $\mathcal{A} \to -1/\mathcal{A}$ for $\phi_{3,1}$ and $\phi_{1,3}$ in this case. Since this transformation does not leave the correlation functions of the original model (in the infinite plane) invariant, it is not a manifestation of a symmetry that has been broken by the boundary. Hence, for these models, fixed and free boundaries are not related via a duality transformation of a self-dual model.

5. Conclusions

Using conformal field theory we have shown how the multicritical Ising universality classes that describe generic $\mathbb{Z}_2$-invariant multicritical points are affected by a boundary. For this purpose, we have derived the order-parameter correlation functions in a semi-infinite plane constrained to fixed and free boundary conditions and obtained the universal surface exponents of the order parameters. In particular, they show that coupling an external field to the order parameter at the boundary is a relevant perturbation only for even-critical models. Our results for one-point amplitudes demonstrate how fixed and free boundaries are related by the Kramers–Wannier duality transformation for the Ising and tricritical Ising models. They also tell us that this relation does not hold for the other multicritical Ising models.

Our correlation functions may also be conformally mapped to an infinite strip or a finite geometry in order to enable a detailed study of boundary- and finite-size effects. In particular, they will govern the susceptibilities, correlation length amplitudes and structure factors of these geometries. In another publication [20], we have carried out the last of these calculations for discs constrained to fixed and free boundary conditions. This was done to visualize how the apparent scaling—induced by finite size and boundaries—of structure factors at intermediate momenta depends on the boundary conditions applied. It was shown that the apparent scaling dimensions that can be extracted from such scaling laws either over- or underestimate the bulk scaling dimensions.

An interesting extension to this work would be to analyse more general boundary conditions. One may, for instance, investigate the effects of a fixed boundary condition that is reversed on half of the boundary. This can be done by inserting a boundary operator at a given point on the boundary [21]. Another interesting possibility from an experimental point of view would be to consider effects of a quenched random field coupling to the order parameter at the boundary [22]. Using the replica method, Cardy showed that such a field is marginal for the Ising model and irrelevant for $x_\parallel > \frac{1}{2}$ [23]. (See also Iglói *et al* in [24].) Hence, for the rest of the models we are considering here, it would not in fact lead to any relevant perturbations. A more general quenched disordered boundary may also be treated using the conformal-invariance technique with boundary operators. So far this has only been carried out for the Ising model [25].

Acknowledgment

The author wishes to thank Henrik Johannesson for valuable discussions.

References

[1] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[2] For a review, see Cardy J L 1987 *Phase Transitions and Critical Phenomena* vol 11 ed C Domb and J L Lebowitz (London: Academic)
[3] Cardy J L 1984 *Nucl. Phys.* B **240** [FS12] 514
[4] Burkhardt T W and Guim I 1991 *J. Phys. A: Math. Gen.* **24** 1557
[5] Zamolodchikov A B 1986 *Sov. J. Nucl. Phys.* **44** 529
[6] Huse D A 1984 *Phys. Rev.* B **30** 3908
[7] Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
[8] Polyakov A M 1970 *JETP Lett.* **12** 381
[9] Dotsenko V S and Fateev V A 1985 *Phys. Lett.* **154B** 291
[10] For a review, see Diehl H W 1986 *Phase Transitions and Critical phenomena* vol 10 ed C Domb and J L Lebowitz (London: Academic)
[11] Burkhardt T W and Cardy J L 1987 *J. Phys. A: Math. Gen.* **20** L233
[12] Fröjdh P and Johannesson H 1991 *Nucl. Phys.* B **366** 429
[13] See for instance Dotsenko V S 1984 *Nucl. Phys.* B **235** [FS11] 54
[14] Cardy J L 1986 *Nucl. Phys.*B **275** [FS17] 200
[15] Balbão D B and Drugowich J R de Felício 1987 *J. Phys. A: Math. Gen.* **20** L207
[16] Alcaraz F C, Baake M, Grimm U and Rittenberg V 1989 *J. Phys. A: Math. Gen.* **22** L5
[17] Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L961
[18] Burkhardt T W 1985 *J. Phys. A: Math. Gen.* **18** L307
[19] Lässig M, Mussardo G and Cardy J L 1991 *Nucl. Phys.* B **348** 591
[20] Fröjdh P 1993 *Phys. Rev.* B **48** 9611
[21] Cardy J L 1989 *Nucl. Phys.* B **324** 581
[22] Diehl H W and Nüsser A 1990 *Z. Phys.* B **79** 69
[23] Cardy J L 1991 *J. Phys. A: Math. Gen.* **24** L1315
[24] Iglói F, Turban L and Berche B 1991 *J. Phys. A: Math. Gen.* **24** L1031
[25] Burkhardt T W and Guim I 1993 *Phys. Rev.* B **47** 14306